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Stability and Steady-State Performance of Hammerstein Spline Adaptive Filter Based on Stochastic Gradient Algorithm

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Abstract: This paper proposes an adaptive step-size approach based on the stochastic gradient-based algorithm for Hammerstein spline adaptive filtering (HSAF).With the good convergence and low complexity computation, a normalized least mean square (NLMS) algorithm is developed with HSAF. A nonlinear Hammerstein adaptive filter consists of memoryless function modified during learning and the spline control point is automatically controlled by gradient-based method. An adaptive step-size approach based on NLMS algorithm with the HSAF is introduced how to derive using the direction of steepest descent method for tracking performance. Hence, the stability and steady-state performance are detailed. The simulation results of trajectories of both update step-size parameters are shown that the initial step-size are reduced, all convergence rates are correspondingly reduced to each own optimum over 100 independent trials of the experiment. Experimental plots of learning curves of mean-squared error of the proposed algorithm for varying the initial step-size parameters are obtained that the proposed algorithm conducts more robust performance and fast convergence compared with the HSAF based on least mean square algorithm, even the large variations of 100-fold initial step-size parameters are used.

Keywords: Hammerstein spline adaptive filtering, Adaptive step-size algorithm, Normalized least mean square algorithm.

1. Introduction

In the recent years, the modelling problem and nonlinear systems identification have been attention that their results used the linear filter are inadequate for system modelling Ref. [1]. In order to solve these problems, a class of nonlinear adaptive filters, named spline adaptive filter (SAF) has been presented in Ref. [1, 2]. SAFs are demonstrated efficiently for identification of nonlinear systems in Ref. [3, 5]. Against the eigen-value spread of the autocorrelation matrix of the input signal in the updating process of SAF algorithm, a class of the stochastic gradient algorithm has been proposed in Ref. [5-7].

Based on SAF architecture, an adaptive and convex combination relied on the properties of adaptive combination of SAF has been proposed in Ref. [8] and SAF architecture extended to many applications Ref. [9-11]. SAF using infinite impulse response for nonlinear system identification Ref. [9] has been conducted. In Ref. [10], the authors have proposed the Wiener spline adaptive filter based on a set-membership normalized least M-estimate algorithm in order to perform fast convergence in the impulsive noise environment. In addition, the SAF approach based on hypothesis testing and variance minimization has been modified for trend analysis in time-series Ref. [11].

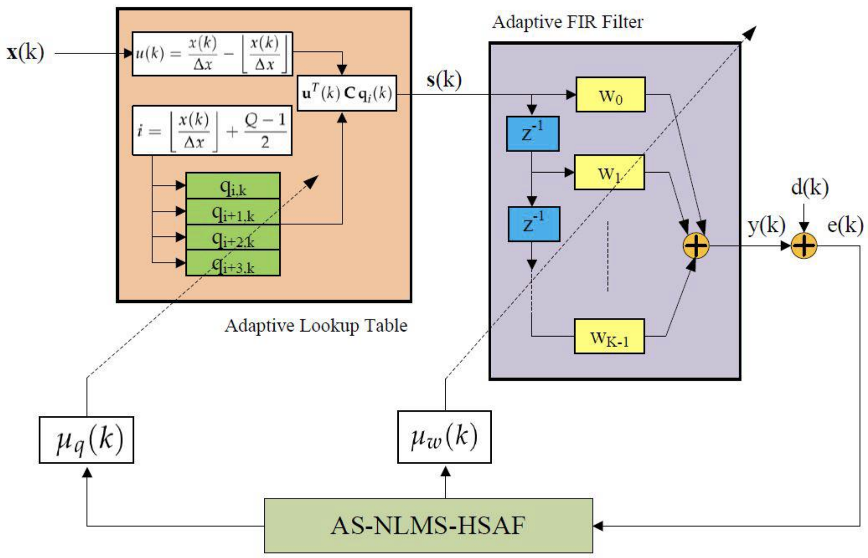

Figure.1 Proposed nonlinear-linear AS-NLMS-HSAF Structure

Hammerstein system is a typical nonlinear model described in many nonlinear systems, as the various system identification of Hammerstein models Ref. [12, 13], an adaptive Hammerstein impedance controller for bionic wrist Ref. [14], the robust Hammerstein adaptive filter using maximum correntropy criterion Ref. [15], and a fuzzy Hammerstein neural network for navigation satellite system Ref. [16]. An unsupervised multistage clustering-based Hammerstein postdistortion approach has presented for visible light communication in Ref. [17].

In contrast, the Hammerstein models have been proposed for system identification in Ref. [18-20]. The multiple model-based Hammerstein parameter varying systems have been studied in the Bayesian approach Ref. [18], the result simulation demonstrated that can work practically. In Ref. [19], the authors have investigated a discrete-time Hammerstein system identification with quantized observations that simulation results of convergence rate are established. In practice, the piecewise continuous Hammerstein systems for identification has been developed in Ref. [20].

A novel nonlinear Hammerstein spline adaptive filter (HSAF) with cubic spline function has been proposed in Ref. [21, 25]. The demonstration of HSAFs orchestrated in the several experimental systems can achieve the good performance as shown in Ref. [22-24]. A single-input single-output fractional-order continuous-time based on Hammerstein-Wiener model Ref. [23] has been implemented for a direct parameter estimation. In Ref. [24], the nonlinear digital cancellation has been modelled the nonlinear power amplifier using Hammerstein spline-based model for impulse response of self-interference channel. HSAF based on the normalized least mean square algorithm is presented in Ref. [25], the results show that algorithm can achieve the good performance compared with the conventional SAF based on least mean square algorithm.

In this paper, we concentrate on the stability and steady-state performance of Hammerstein spline adaptive filtering based on normalized least mean square algorithm (NLMS-HSAF). A nonlinear Hammerstein adaptive filter consists of memoryless property function modified between learning and the spline control point is an automatically controlled by gradient-based algorithm. An adaptive step-size mechanism is to induce with NLMS-HSAF algorithm using steepest descent method for tracking performance. In particular, we emphasize a robust performance, computational analysis and convergence properties of proposed adaptive step-size approach based on NLMS-HSAF algorithm in detail.

We organize this paper in the following. Section 2 introduces the Hammerstein spline adaptive filter based on least mean square algorithm. Section 3 describes the proposed Hammerstein cubic spline adaptive filter based on adaptive step-size normalized least mean square algorithm. Section 4 presents the stability and steady-state performance and section 5 shows the simulation results and discussion. Finally, section 6 concludes the work.

Notations are used through this paper. Operator $(\cdot)^T$ indicates the transposition operation. Matrices and vectors are in bold uppercase and lowercase, respectively.

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2. Hammerstein spline adaptive filtering

Following Ref. [21], the Hammerstein spline adaptive filter (HSAF) is called nonlinear-linear network, as shown in Figure 1. This network consists of the nonlinear and linear network. In the first part, the HSAF uses for the identification of Hammerstein nonlinear systems Ref. [3] combined with an adaptive look-up table Ref. [4] with an adaptive control points vectors and spline interpolation function to the nonlinear network. For the second part, an adaptive approach is modified to linear coefficient filter. Both the adaptive nonlinear and linear filters are to introduce with the minimization cost function.

Consider an error $e(k)$ as

$$e(k) = d(k) - y(k) = d(k) - \boldsymbol{w}^T(k)\boldsymbol{s}(k), \quad (1)$$

where $y(k)$ is the Hammerstein spline adaptive filtering (HSAF) output, $d(k)$ is the desired signal and $\mathbf{w}(k)$ is the adaptive coefficient tap-weight vector.

The output of nonlinearity $\mathbf{s}(k)$ is given Ref. [1]

$$\boldsymbol{s}(k) = \boldsymbol{u}^T(k)\boldsymbol{C}\boldsymbol{q}_i(k), \quad (2)$$

$$\boldsymbol{u}(k) = [u^3(k),\ u^2(k),\ u(k),\ 1]^T, \quad (3)$$

where the nonlinearity output vector $\boldsymbol{s}(k)$ is referred to a nonlinear function with the span index i and the local parameter $u(k)$, where $u(k) \in [0,1]$. Since, the parameter $\boldsymbol{C}$ is the spline basis matrix.

The control point tap-weight vector $\boldsymbol{q}_i(k)$ is given as

$$\boldsymbol{q}_i(k) = [q_i(k),\ q_{i+1}(k),\ q_{i+2}(k),\ q_{i+3}(k)]^T. \quad (4)$$

Following Ref. [1], the local parameter $u(k)$ and index parameter i can be evaluated as

$$u(k) = \frac{x(k)}{\Delta x} - \left\lfloor \frac{x(k)}{\Delta x} \right\rfloor, \quad (5)$$

$$i = \left\lfloor \frac{x(k)}{\Delta x} \right\rfloor + \frac{Q-1}{2}, \quad (6)$$

where Δx is the uniform space between two of adjacent control points and $x(k)$ is the input vector with the length of tap delay N. The constant parameter Q is the number of control point and operator $\lfloor \cdot \rfloor$ defines as a floor operator.

We can obtain the minimized cost function based on the least mean square algorithm for Hammerstein spline adaptive filter (LMS-HSAF) as

$$J(\boldsymbol{w}(k), \boldsymbol{q}_i(k)) = \min_{\boldsymbol{w},\boldsymbol{q}} \left\{ \frac{1}{2} |e(k)|^2 \right\}, \quad (7)$$

where $e(k)$ is defined in Eq. (1).

Hence, the proposed coefficient tap-weight vectors $\mathbf{w}(k)$ and $\mathbf{q}_i$(k) of LMS-HSAF algorithm can be orchestrated by

$$\boldsymbol{w}(k+1) = \boldsymbol{w}(k) - \mu_w(k)\{\nabla_w J(\boldsymbol{w}(k), \boldsymbol{q}_i(k))\}, (8)$$

$$\boldsymbol{q}_i(k+1) = \boldsymbol{q}_i(k) - \mu_q(k)\{\nabla_q J(\boldsymbol{w}(k), \boldsymbol{q}_i(k))\}, (9)$$

where μ_w and μ_q are the step-size parameters for learning rate of $\mathbf{w}(k)$ and $\mathbf{q}_i(k)$ of HSAF structure.

Following the chain rule Ref. [21] by differentiating the cost function in Eq. (7) with respect to (w.r.t.) $\mathbf{w}(k)$ and $\mathbf{q}_i(k)$, we arrive at

$$\nabla_w J(\boldsymbol{w}(k), \boldsymbol{q}_i(k)) = \frac{\partial J(\boldsymbol{w}(k), \boldsymbol{q}_i(k))}{\partial \boldsymbol{w}(k)} = -\{e(k)\boldsymbol{s}(k)\}, \quad (10)$$

$$\nabla_q J(\boldsymbol{w}(k), \boldsymbol{q}_i(k)) = \frac{\partial J(\boldsymbol{w}(k), \boldsymbol{q}_i(k))}{\partial \boldsymbol{w}(k)} = -\{\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)e(k)\}, \quad (11)$$

Substituting Eqs. (10) and (11) in Eqs. (8) and (9), the coefficient tap-weight vectors $\boldsymbol{w}(k)$ and $\boldsymbol{q}_i(k)$ are given by

$$\therefore \boldsymbol{w}(k+1) = \boldsymbol{w}(k) + \mu_w \boldsymbol{s}(k)e(k), \quad (12)$$

$$\therefore \boldsymbol{q}_i(k+1) = \boldsymbol{q}_i(k) + \mu_q \boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)e(k), \quad (13)$$

where $e_n, \boldsymbol{s}_n$ and $\boldsymbol{u}_n$ are defined in Eqs. (1), (2) and (3), respectively.

3. Proposed Hammerstein cubic spline adaptive filtering based on adaptive step-size normalized least mean square algorithm

In addition to the normalized least mean square (NLMS) algorithm, this proposed modification makes an effort to improve the properties of the least mean square (LMS) algorithm. In this section, we provide an overview of proposed NLMS modification based on HSAF, including the adaptive step-size mechanism for dealing with a trade-off between the rate of convergence and excess mean square error (EMSE).

Following Ref. [25], the minimized cost function of proposed adaptive step-size normalized least mean

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square for Hammerstein cubic spline adaptive filter (AS-NLMS- HSAF) can be expressed as

$$\tilde{J}\big(\boldsymbol{w}(k), \boldsymbol{q}_i(k)\big) = \min_{\boldsymbol{w},\boldsymbol{q}} \left\{\frac{1}{2}\frac{|e(k)|^2}{\|\boldsymbol{u}(k)\|^2}\right\}, \tag{14}$$

where $e(k)$ is defined in (1).

Following the chain rule by differentiating the cost function in Eq. (14) w.r.t. $\mathbf{w}(k)$, we arrive at

$$\nabla_{\boldsymbol{w}}\tilde{J} = \frac{\partial \tilde{J}\big(\boldsymbol{w}(k),\boldsymbol{q}_i(k)\big)}{\partial \boldsymbol{w}(k)} = -\left\{\frac{e(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}(k)\|^2}\right\}. \tag{15}$$

We define the derivative of cost function in Eq. (14) w.r.t. $\mathbf{q}_i(k)$ by the chain rule as

$$\nabla_{\boldsymbol{q}}\tilde{J} = \frac{\partial \tilde{J}\big(\boldsymbol{w}(k),\boldsymbol{q}_i(k)\big)}{\partial \boldsymbol{q}_i(k)} = -\left\{\frac{\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)e(k)}{\|\boldsymbol{u}(k)\|^2}\right\}. \tag{16}$$

Hence, the proposed coefficient tap-weight vector $\mathbf{w}(k)$ of AS-NLMS-HSAF algorithm can be orchestrated by

$$\boldsymbol{w}(k+1) = \boldsymbol{w}(k) - \mu_w(k)\{\nabla_{\boldsymbol{w}}\tilde{J}\}, \tag{17}$$

where $\mu_{\mathrm{w}}(k)$ is an adaptive step-size parameter for learning rate of $\boldsymbol{w}(k)$ of HSAF structure.

Substituting Eq. (15) in Eq. (17), the coefficient tap-weight vector $\mathbf{w}(k)$ is given by

$$\therefore \boldsymbol{w}(k+1) = \boldsymbol{w}(k) + \mu_w(k)\frac{\boldsymbol{s}(k)e(k)}{\|\boldsymbol{u}(k)\|^2}, \tag{18}$$

where $e_n, \boldsymbol{s}_n$ and $\boldsymbol{u}_n$ are defined in Eq. (1), (2) and (3), respectively.

Similarly, the proposed control points vector $\boldsymbol{q}_i$(k) of AS-NLMS-HSAF can be implemented as

$$\boldsymbol{q}_i(k+1) = \boldsymbol{q}_i(k) - \mu_q(k)\{\nabla_{\boldsymbol{q}}\tilde{J}\}, \tag{19}$$

$$\therefore \boldsymbol{q}_i(k+1) = \boldsymbol{q}_i(k) + \mu_q(k)\frac{\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)e(k)}{\|\boldsymbol{u}(k)\|^2}, \tag{20}$$

where $\mu_q(k)$ is an adaptive step-size parameter for learning rate of $\mathbf{q}_i(k)$ of HSAF structure.

3.1 Adaptive step-size mechanism

An idea of adaptive step-size mechanism is to select between the convergence rate, EMSE and the ability of filters for tracking signals Ref. [26-27]. At the beginning, the tap-weight vectors $\mathbf{w}(k)$ and $\mathbf{q}_i(k)$ are far from optimal, so the step-size parameters $\mu_{\mathrm{w}}(k)$ and $\mu_q(k)$ should be large to make tap-weight vectors $\mathbf{w}(k)$ and $\mathbf{q}_i(k)$ close to the desired values. Meanwhile, the mean square error (MSE) of filters start converging to the steady-state, the step-size parameters $\mu_{\mathrm{w}}(k)$ and $\mu_q(k)$ should be decreased to reduce the EMSE.

We determine an adaptive step-size mechanism based on NLMS-HSAF algorithm using the direction of steepest descent method. Considering the derivative of cost function in Eq. (14) w.r.t. $\mu_{\mathrm{w}}(k)$, that is

$$\nabla\tilde{J}_{\mu_w} = \frac{\partial}{\partial\mu_w(k)}\frac{1}{2}\left\{\frac{e^2(k)}{\|\boldsymbol{u}(k)\|^2}\right\} = \left\{\frac{e(k)\boldsymbol{s}^T(k)\boldsymbol{\rho}_w(k)}{\|\boldsymbol{u}(k)\|^2}\right\}, \tag{21}$$

where $\boldsymbol{\rho}_{\mathrm{w}}(k)$ is the derivative of $\boldsymbol{w}(k)$ w.r.t. $\mu_W(k)$, as $\boldsymbol{\rho}_{\mathrm{w}}(k) = \frac{\partial \mathbf{w}(k)}{\partial \mu_{\mathrm{w}}(k)}$.

Let us define the derivative of $\mathbf{w}(k)$ w.r.t. $\mu_{\mathrm{w}}(k)$ as

$$\begin{aligned}\frac{\partial \boldsymbol{w}(k+1)}{\partial\mu_w(k)} &= \frac{\partial \boldsymbol{w}(k)}{\partial\mu_w(k)} - \frac{\partial}{\partial\mu_w(k)}\left\{\frac{\mu_w(k)e(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}(k)\|^2}\right\}\\ &= \frac{\partial \boldsymbol{w}(k)}{\partial\mu_w(k)} - \left(\frac{\mu_w(k)\boldsymbol{s}^T(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}(k)\|^2}\cdot\frac{\partial \boldsymbol{w}(k)}{\partial\mu_w(k)} + \frac{e(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}(k)\|^2}\right).\end{aligned} \tag{22}$$

Since, $\boldsymbol{\rho}_{\mathrm{w}}(k)$ as the derivative of $\mathbf{w}(k)$ w.r.t. $\mu_{\mathrm{w}}(k)$ in Eq. (22) can be rewritten as

$$\begin{aligned}&\boldsymbol{\rho}_{\boldsymbol{w}}(k+1)\\ &= \boldsymbol{\rho}_{\boldsymbol{w}}(k) - \frac{\mu_w(k)\boldsymbol{s}^T(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}(k)\|^2}\boldsymbol{\rho}_{\boldsymbol{w}}(k) - \frac{\boldsymbol{s}(k)e(k)}{\|\boldsymbol{u}(k)\|^2}.\end{aligned} \tag{23}$$

Therefore, we can obtain an adaptive step-size $\mu_{\mathrm{w}}(k)$ of linear network part $\mathbf{w}(k)$ of AS-NLMS-HSAF with the gradient points of the direction of steepest descent points in the negative gradient direction as

$$\mu_w(k+1) = \mu_w(k) - \alpha_w\big(\nabla J_{\mu_w}\big)$$

$$\therefore \mu_w(k+1) = \mu_w(k) - \alpha_w\frac{\boldsymbol{s}^T(k)\boldsymbol{\rho}_w(k)e(k)}{\|\boldsymbol{u}(k)\|^2} \tag{24}$$

where $0 < \alpha_w < 1$ and $\boldsymbol{\rho}_{\mathrm{w}}(k)$ is given in Eq. (23).

In a similar way, we specify an adaptive step-size $\mu_q(k)$approach based on AS-NLMS-HSAF algorithm. Hence, the derivative of cost function in Eq. (14) w.r.t. $\mu_q(k)$ is computed as

$$\nabla\tilde{J}_{\mu_q} = \frac{\partial}{\partial\mu_q(k)}\frac{1}{2}\left\{\frac{e^2(k)}{\|\boldsymbol{u}(k)\|^2}\right\} = \left\{\frac{e(k)\boldsymbol{w}(k)\boldsymbol{u}^T(k)\boldsymbol{C}\,\boldsymbol{\rho}_q(k)}{\|\boldsymbol{u}(k)\|^2}\right\}, \tag{25}$$

Algorithm 1. Hammerstein cubic spline adaptive filtering based on adaptive step-size normalized least mean square (AS-NLMS-HSAF) proposed

```
CREATE CONSTANT: Δx, Q, α_w, α_q
INITIAL: w(0), q(0), ρ_q(0), μ_w(0), μ_q(0), C = the spline basis matrix
FOR (n = 0; n < (N − 1); n++)
```

$$\begin{aligned}
u(k) &= \frac{x(k)}{\Delta x} - \left\lfloor \frac{x(k)}{\Delta x} \right\rfloor \\
i &= \left\lfloor \frac{x(k)}{\Delta x} \right\rfloor + \frac{Q-1}{2} \\
\mathbf{u}(k) &= [u^3(k), u^2(k), u(k), 1]^T \\
\mathbf{q}_i(k) &= [q_i(k), q_{i+1}(k), q_{i+2}(k), q_{i+3}(k)]^T \\
\boldsymbol{s}(k) &= \boldsymbol{u}^T(k)\boldsymbol{C}\mathbf{q}_i(k) \\
e(k) &= d(k) - \boldsymbol{w}^T(k)\boldsymbol{s}(k) \\
\boldsymbol{w}(k+1) &= \boldsymbol{w}(k) + \frac{\mu_w(k)\boldsymbol{s}(k)e(k)}{\|\boldsymbol{u}(k)\|^2} \\
\mu_w(k+1) &= \mu_w(k) - \alpha_w \frac{\boldsymbol{s}^T(k)\boldsymbol{\rho_w}(k)e(k)}{\|\boldsymbol{u}(k)\|^2} \\
\boldsymbol{\rho}_{\mathbf{w}}(k+1) &= \boldsymbol{\rho_w}(k) - \frac{\mu_w(k)\boldsymbol{s}^T(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}(k)\|^2}\boldsymbol{\rho_w}(k) - \frac{\boldsymbol{s}(k)e(k)}{\|\boldsymbol{u}(k)\|^2} \\
\mathbf{q}_i(k+1) &= \mathbf{q}_i(k) + \frac{\mu_q(k)\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)e(k)}{\|\boldsymbol{u}(k)\|^2} \\
\mu_q(k+1) &= \mu_q(k) - \alpha_q \frac{v^T(k)\boldsymbol{\rho_q}(k)e(k)}{\|\boldsymbol{u}(k)\|^2} \\
\boldsymbol{\rho}_{\mathbf{q}}(k+1) &= \boldsymbol{\rho}_{\mathbf{q}}(k) - \frac{\mu_q(k)v^T(k)v(k)}{\|\boldsymbol{u}(k)\|^2}\boldsymbol{\rho}_{\mathbf{q}}(k) + \frac{v(k)e(k)}{\|\boldsymbol{u}(k)\|^2} \\
v(k) &= \boldsymbol{u}^T(k)\boldsymbol{C}\boldsymbol{w}(k)
\end{aligned}$$

```
END
```

where $\boldsymbol{\rho_q}(k)$ is the derivative of $\boldsymbol{q}_i(k)$ w.r.t. $\mu_q(k)$, as $\boldsymbol{\rho_q}(k) = \frac{\partial \boldsymbol{q}_i(k)}{\partial \mu_q}$ and *a priori* error of system $e(k)$, can be rewritten in forms of $\boldsymbol{q}_i(k)$ as

$$\begin{aligned} e(k) &= d(k) - \boldsymbol{w}^T(k)\boldsymbol{s}(k) \\ &= d(k) - \boldsymbol{w}^T(k)\big(\boldsymbol{u}(k)\boldsymbol{C}^T(k)\boldsymbol{q_i}(k)\big), \end{aligned} \quad (26)$$

Let us determine the derivative of $\mathbf{q}_i(k)$ w.r.t. $\mu_q(k)$ as

$$\begin{aligned} \frac{\partial \boldsymbol{q}_i(k+1)}{\partial \mu_q(k)} &= \frac{\partial \boldsymbol{q}_i(k)}{\partial \mu_q(k)} + \frac{\partial}{\partial \mu_q(k)}\left\{\frac{\mu_q(k)\boldsymbol{u}(k)\boldsymbol{C}\boldsymbol{w}(k)e(k)}{\|\boldsymbol{u}(k)\|^2}\right\} \\ &= \frac{\partial \boldsymbol{q}_i(k)}{\partial \mu_q(k)} + \frac{\mu_q(k)\big(\boldsymbol{C}^T\boldsymbol{u}(k)\boldsymbol{w}(k)\big)\big(\boldsymbol{w}(k)\boldsymbol{u}^T(k)\boldsymbol{C}\big)}{\|\boldsymbol{u}(k)\|^2} \\ &\quad \cdot \frac{\partial \boldsymbol{q}_i(k)}{\partial \mu_q(k)} + \frac{\boldsymbol{u}^T(k)\boldsymbol{C}\boldsymbol{w}(k)e(k)}{\|\boldsymbol{u}(k)\|^2}. \end{aligned} \quad (27)$$

Hence, $\boldsymbol{\rho}_{\mathbf{q}}(k)$ as the derivative of $\mathbf{q}_i(k)$ w.r.t. $\mu_q(k)$ in Eq. (27) can be organized as

$$\begin{aligned} &\boldsymbol{\rho_q}(k+1) \\ &= \boldsymbol{\rho_q}(k) - \frac{\mu_q(k)v^T(k)v(k)}{\|\boldsymbol{u}(k)\|^2}\boldsymbol{\rho_q}(k) + \frac{v(k)e(k)}{\|\boldsymbol{u}(k)\|^2}, \end{aligned} \quad (28)$$

where $v(k)$ is given by

$$v(k) = \boldsymbol{u}^T(k)\boldsymbol{C}\boldsymbol{w}(k). \quad (29)$$

Therefore, an adaptive step-size $\mu_q(k)$ of nonlinear network part $\mathbf{q}_i(k)$ of AS-NLMS-HSAF is computed similarly as

$$\begin{aligned} \mu_q(k+1) &= \mu_q(k) - \alpha_q\left(\nabla \tilde{J}_{\mu_q}\right) \\ \therefore \mu_q(k+1) &= \mu_q(k) - \alpha_q \frac{v^T(k)\boldsymbol{\rho_q}(k)e(k)}{\|\boldsymbol{u}(k)\|^2}, \end{aligned} \quad (30)$$

where $0 < \alpha_q < 1$ and $\boldsymbol{\rho}_{\mathbf{q}}(k)$ is given in Eq. (28).

A summary of proposed AS-NLMS-HSAF algorithm shows in Algorithm 1.

4. Stability and steady-state performance

We address the point of stability and steady-state performance of adaptive step-size normalized least mean square algorithm based on Hammerstein cubic spline adaptive filtering (AS-NLMS-HSAF). The goals of adaptive filters $\mathbf{w}(k)$ and $\mathbf{q}_i(k)$ are to track the optimum filters $\mathbf{w}_{opt}(k)$ and $\mathbf{q}_{opt}(k)$ as fast as possible.

Since, we can investigate the performance by measuring the deviation of the coefficient tap-weight vectors as

$$\Delta\boldsymbol{w}(k+1) = \boldsymbol{w}(k) - \boldsymbol{w}_{opt}(k), \tag{31}$$

$$\Delta\boldsymbol{q}(k+1) = \boldsymbol{q}_i(k) - \boldsymbol{q}_{opt}(k). \tag{32}$$

Performance of adaptive filters is primarily measured by evaluating the value of the mean square error (MSE). Hence, measuring the deviation of adaptive filters from the corresponding optimum filters can be determined to investigate their performance.

Consequently, the mean square deviation (MSD) of coefficient vector $\mathbf{w}(k)$ as

$$\begin{aligned}\mathfrak{D}_w(k) &= E\left\{\left\|\boldsymbol{w}(k) - \boldsymbol{w}_{opt}(k)\right\|^2\right\}\\ &= E\{\|\Delta\boldsymbol{w}(k+1)\|^2\},\end{aligned} \tag{33}$$

measures the distance between the coefficient vectors of adaptive $\mathbf{w}(k)$ and optimum $\mathbf{w}_{opt}(k)$ filters. $E\{\cdot\}$ is an expectation operator and $\|\cdot\|$ defines as an operator norm.

In a similar way, the MSD of coefficient vector $\mathbf{q}_i(k)$ as

$$\begin{aligned}\mathfrak{D}_q(k) &= E\left\{\left\|\boldsymbol{q}_i(k) - \boldsymbol{q}_{opt}(k)\right\|^2\right\}\\ &= E\{\|\Delta\boldsymbol{q}(k+1)\|^2\},\end{aligned} \tag{34}$$

calculates the distance between $\mathbf{q}_i(k)$ and optimum filters $\boldsymbol{q}_{\mathrm{opt}}(k)$.

4.1 Stability

We consider the convergence properties of adaptive coefficient vector $\mathbf{w}(k)$ by using the Taylor series expansion Ref. [5] of the first order of differentiating error $e_w(k)$ and deviation of $\mathbf{w}(k)$ as

$$\begin{aligned}e_w(k+1) &\approx e_w(k) + \frac{\partial e(k)}{\partial \boldsymbol{w}(k)}\Delta\boldsymbol{w}(k)\\ &= e_w(k) - \mu_w(k)\frac{\boldsymbol{s}^T(k)\boldsymbol{s}(k)e_w(k)}{\|\boldsymbol{u}^T(k)\|^2}\\ &= e_w(k)\left\{1 - \mu_w(k)\frac{\boldsymbol{s}^T(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}^T(k)\|^2}\right\}.\end{aligned} \tag{35}$$

Then, we assume the $e_w(k+1) < e_w(k)$ in Eq. (35), that is

$$\left|1 - \mu_w(k)\frac{\boldsymbol{s}^T(k)\boldsymbol{s}(k)}{\|\boldsymbol{u}^T(k)\|^2}\right| < 1. \tag{36}$$

Clearly, the learning rate of adaptive step-size referring to coefficient vector $\mathbf{w}(k)$ is simplified as

$$\therefore 0 < \mu_w(k) < \frac{2\|\boldsymbol{u}^T(k)\|^2}{\boldsymbol{s}^T(k)\boldsymbol{s}(k)}. \tag{37}$$

According to the nonlinear and nonstationary input signals, an adaptive regularized term $\delta_w(\mathrm{k})$ of $\mathbf{w}(k)$ can be added in the denominator of Eq. (37). Then, the learning rate with the regularized term is given as

$$0 < \mu_w(k) < \frac{2\|\boldsymbol{u}^T(k)\|^2}{\boldsymbol{s}^T(k)\boldsymbol{s}(k) + \delta_w(k)}. \tag{38}$$

Following Ref. [3], the adaptive regularized term $\delta_w(\mathrm{k})$ of $\mathbf{w}(k)$ can be expressed as

$$\delta_w(k+1) = \delta_w(k) - \eta_w\left(\nabla_{\delta_w}\tilde{J}\right), \tag{39}$$

where $\eta_w(k)$ is the learning rate referring to $\mathbf{w}(k)$.

Differentiating $\tilde{J}(\mathbf{w}(k), \mathbf{q}_i(k))$ in Eq. (14) w.r.t. $\delta_w(\mathrm{k})$ by using the chain rule, we get

$$\begin{aligned}\nabla_{\delta_w}\tilde{J} &= \frac{\partial\tilde{J}(\boldsymbol{w}(k),\boldsymbol{q}_i(k))}{\partial\delta_w(k)}\\ &= \frac{\partial\tilde{J}(\boldsymbol{w}(k),\boldsymbol{q}_i(k))}{\partial e(k)}\cdot\frac{\partial e(k)}{\partial y(k)}\cdot\frac{\partial y(k)}{\partial \boldsymbol{w}(k)}\cdot\frac{\partial \boldsymbol{w}(k)}{\partial \mu_w(k)}\cdot\frac{\partial \mu_w(k)}{\partial \delta(k)}.\\ &= \frac{4e(k)e(k-1)\boldsymbol{s}^T(k)\boldsymbol{s}(k-1)}{[\boldsymbol{s}^T(k)\boldsymbol{s}(k-1)+\delta_w(k-1)]^2}.\end{aligned} \tag{40}$$

Therefore, we obtain the proposed adaptive regularized term $\delta_w(k)$ of $\mathbf{w}(k)$ as

$$\therefore \delta_w(k+1) = \delta_w(k) - \eta_w\frac{e(k)e(k-1)\boldsymbol{s}^T(k)\boldsymbol{s}(k-1)}{[\boldsymbol{s}^T(k)\boldsymbol{s}(k-1)+\delta_w(k-1)]^2}. \tag{41}$$

Since, we develop the convergence properties of adaptive coefficient vector $\mathbf{q}_i(k)$ using the Taylor series expansion Ref. [5] of the first order of differentiating error $\mathbf{q}_i(k)$ as

$$\begin{aligned}e_q(k+1) &\approx e_q(k) + \frac{\partial e(k)}{\partial \boldsymbol{q}_i(k)}\Delta\boldsymbol{q}_i(k)\\ &= e_q(k) - \mu_q(k)\frac{v^T(k)\boldsymbol{w}(k)v(k)e_q(k)}{\|\boldsymbol{u}^T(k)\|^2}\\ &= e_q(k)\left\{1 - \mu_q(k)\frac{v^T(k)\boldsymbol{w}(k)v(k)}{\|\boldsymbol{u}^T(k)\|^2}\right\}.\end{aligned} \tag{42}$$

So, we assume the $e_q(k+1) < e_q(k)$ in Eq. (42), that is

$$\left|1 - \mu_q(k)\frac{v^T(k)\boldsymbol{w}(k)v(k)}{\|\boldsymbol{u}^T(k)\|^2}\right| < 1. \tag{43}$$

Obviously, the learning rate of adaptive step-size $\mu_q(k)$ of coefficient vector $\boldsymbol{q}_i(k)$ is determined as

$$\therefore 0 < \mu_q(k) < \frac{2\|\boldsymbol{u}^T(k)\|^2}{v^T(k)\boldsymbol{w}(k)v(k)}. \quad (44)$$

Imposing on the learning rates in Eqs. (38) and (44), the restrictive constraint must be satisfied by using $\mu_w(k)$ and $\mu_q(k)$. Hence, we rewrite the output error expansion as

$$\begin{aligned} e(k+1) &= e(k) + \Delta\boldsymbol{w}(k) \cdot \frac{\partial e(k)}{\partial \boldsymbol{w}(k)}\bigg|_{\boldsymbol{q}_i(k)=const} \\ &\quad + \Delta\boldsymbol{q}_i(k) \cdot \frac{\partial e(k)}{\partial \boldsymbol{q}_i(k)}\bigg|_{\boldsymbol{w}(k)=const} \\ &= e(k) - \mu_{\mathfrak{w}}(k)\frac{\|\boldsymbol{s}(k)\|^2 e(k)}{\|\boldsymbol{u}(k)\|^2}. \\ &\quad - \mu_q(k)\frac{\|\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)\|^2 e(k)}{\|\boldsymbol{u}(k)\|^2}. \\ &= e(k) - \left[1 - \mu_{\mathfrak{w}}(k)\frac{\|\boldsymbol{s}(k)\|^2}{\|\boldsymbol{u}(k)\|^2} \right. \\ &\quad \left. - \mu_q(k)\frac{\|\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)\|^2}{\|\boldsymbol{u}(k)\|^2}\right]. \end{aligned} \quad (45)$$

By using the condition that $|e(k+1)| < |e(k)|$, we obtain the constraint as

$$\begin{aligned} 1 + \mu_w(k)\frac{\|\boldsymbol{s}(k)\|^2}{\|\boldsymbol{u}(k)\|^2} + \mu_q(k)\frac{\|\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)\|^2}{\|\boldsymbol{u}(k)\|^2} < 1 \\ \therefore 0 < \mu_w(k)\frac{\|\boldsymbol{s}(k)\|^2}{\|\boldsymbol{u}(k)\|^2} + \mu_q(k)\frac{\|\boldsymbol{u}(k)\boldsymbol{C}^T\boldsymbol{w}(k)\|^2}{\|\boldsymbol{u}(k)\|^2} < 2\,. \end{aligned} \quad (46)$$

4.2 Mean square error performance

In this section, we investigate the mean square error (MSE) performance of the proposed AS-NLMS-HSAF structure at the steady-state. Following Ref. [28], the MSE can be decomposed as

$$J_{MSE_w}(k) = E\{\|\epsilon_w(k)\|^2\} = J_{MMSE_w}(k) + J_{EX_w}(k), \quad (47)$$

where $J_{MMSE_w}(k)$ is the minimum mean square error (MMSE) given by

$$J_{MMSE_w}(k) = E\left\{\left\|\epsilon_{opt,w}(k)\right\|^2\right\}, \quad (48)$$

where $\epsilon_{opt,w}(k)$ is the *a posteriori* optimum filtering error of $\mathbf{w}(k)$ as

$$\epsilon_{opt,w}(k) = d(k) - \boldsymbol{w}_{opt}{}^T(k)\boldsymbol{s}(k). \quad (49)$$

Obviously, the *a posteriori* excess mean square error (EMSE) $J_{EX_w}(k)$ is given by

$$J_{EX_w}(k) = J_{MSE_w}(k) - J_{MMSE_w}(k). \quad (50)$$

For *a priori* EMSE $J'_{EX_w}(k)$, we can express as

$$J'_{EX_w}(k) = J'_{MSE_w}(k) - J'_{MMSE_w}(k), \quad (51)$$

where

$$J'_{MSE_w}(k) = E\{\|e_w(k)\|^2\}, \quad (52)$$

$$J'_{MMSE_w}(k) = E\left\{\left\|e_{opt,w}(k)\right\|^2\right\}, \quad (53)$$

with

$$e_{opt,w}(k) = d(k) - \boldsymbol{w}_{opt}{}^T(k)\boldsymbol{s}(k), \quad (54)$$

as the *a priori* optimum filtering error of $\mathbf{w}(k)$.
ASSUMPTION I: We assume the adaptive filters working in a stationary signal operated environment, we have

$$\begin{aligned} E\{\epsilon_{opt,w}(k)\} \approx e_{opt,w}(k), \\ E\{\epsilon_{opt,q}(k)\} \approx e_{opt,q}(k), \end{aligned}$$

that the *a priori* and *a posteriori* optimum errors are identical.

ASSUMPTION II: We consider the convergence condition, that are of

$$\begin{aligned} &E\{\epsilon_{opt,w}(k)\} \to 0,\ as\ k \to \infty, \\ &E\{\boldsymbol{w}(k)\} \to \boldsymbol{w}_{opt}(k),\ as\ k \to \infty, \\ &E\{\epsilon_{opt,q}(k)\} \to 0,\ as\ k \to \infty, \\ &E\{\boldsymbol{q_i}(k)\} \to \boldsymbol{q}_{opt}(k),\ as\ k \to \infty. \end{aligned}$$

By using *ASSUMPTION I*, the MSE of $\mathbf{q}_i(k)$ can be expressed as

$$J_{MSE_q}(k) = E\left\{\left\|\epsilon_q(k)\right\|^2\right\} = J_{MMSE_q}(k) + J_{EX_q}(k), \quad (55)$$

where $J_{MMSE_q}(k)$ is the minimum mean square error (MMSE) of $\mathbf{q}_i(k)$ using *ASSUMPTION II* given by

$$J_{MMSE_q}(k) = E\left\{\left\|\epsilon_{opt,q}(k)\right\|^2\right\}, \quad (56)$$

where $\epsilon_{opt,q}(k)$ is the *a posteriori* optimum filtering error of $\mathbf{q}_i(k)$ as

$$\begin{aligned}\epsilon_{opt,q}(k) &= d(k) - \boldsymbol{w}(k)\boldsymbol{u}^T(k)\boldsymbol{\mathcal{C}}\boldsymbol{q}_{opt}(k) \\ &= d(k) - v^T(k)\boldsymbol{q}_{opt}(k), \end{aligned} \quad (57)$$

where $v(k)$ is given in Eq. (29).

Similarly, the *a posteriori* EMSE $J_{EX_q}(k)$ is given by

$$J_{EX_q}(k) = J_{MSE_q}(k) - J_{MMSE_q}(k). \quad (58)$$

Therefore, the *a priori* EMSE $J'_{EX_q}(k)$ of $\mathbf{q}_i(k)$ can be evaluated as

$$J'_{EX_q}(k) = J'_{MSE_q}(k) - J'_{MMSE_q}(k). \quad (59)$$

where

$$\begin{aligned} J'_{MSE_q}(k) &= E\left\{\left\|e_q(k)\right\|^2\right\}, \\ J'_{MMSE_q}(k) &= E\left\{\left\|e_{opt,q}(k)\right\|^2\right\}, \end{aligned} \quad (60)$$

with

$$e_{opt,q}(k) = d(k) - v^T(k)\boldsymbol{q}_{opt}(k), \quad (61)$$

as the *a priori* optimum filtering error of $\mathbf{q}_i(k)$.

5. Simulation experiments

In this simulation experiments, we consider the random process for input colour signal that is also generated by this equation

$$x(k) = \alpha \cdot x(k) + \sqrt{1+\alpha^2} \cdot \xi(k), \quad (62)$$

where $\xi(k)$ is unit variance and zero mean white Gaussian noise and α = [0.01, 0.99].

With a two-coefficient LMS-based adaptive filters as update coefficient FIR filter $\mathbf{w}(k)$ and adaptive control points LUT vector $\mathbf{q}_i(k)$, the performance of proposed AS-NLMS-HSAF algorithm is evaluated with LMS-HSAF Ref. [21] algorithm over 20,000 samples and 100 Monte Carlo trials. In the system identification, an unknown system consists of a linear component and a nonlinear memoryless function using a 23-point of LUT interpolated by cubic spine function as detailed in [1]. We refer the readers to Ref. [1, 3] for more details.

Initial parameters for both AS-NLMS-HSAF and LMS-HSAF algorithms are summarized in the following: $\mathbf{w}(0) = \mathbf{q}(0) = 0.01 \cdot [100\ldots]^T$, SNR = 40dB, an interval sampling Δx = 0.2 Ref. [9] and tap length M = 7.

We obtain the learning curves of mean square error (MSE) with α = 0.85 shown in Fig.2 and with α = 0.15 illustrated in Fig. 3, respectively. Comparing these results, we note that the learning curves of proposed AS-NLMS-HSAF are similar and fast convergence with varying large initial step-size parameters compared with the fixed step-size parameter of LMS-HSAF Ref. [21]. According to Fig.2 and Fig. 3, we observe a property of LMS algorithm from these curves is that, the convergence of adaptive algorithm to its steady-state value is slower meanwhile the steady-state MSE is smaller, when the step-size is declined.

Trajectories of adaptive step-size parameters $\mu_{\mathrm{w}}(k)$ of adaptive coefficient FIR filter and $\mu_{\mathrm{q}}(k)$ of update control points vector of proposed AS-NLMS-HSAF are illustrated with varying initial step-size setting at α = 0.15 in Fig. 4 and Fig. 5, respectively. We remark that both of trajectories of $\mu_{\mathrm{w}}(k)$ and $\mu_{\mathrm{q}}(k)$ appear to converge into the steady-state with initial variation.

6. Conclusion

A Hammerstein structure in forms of cubic spline adaptive filtering has been proposed with adaptive step-size approach using the steepest descent scheme for both the coefficient FIR filter and control points vector of HSAF architecture. The proposed HSAF-based algorithm using normalized version of LMS has been introduced how to derive with the method of adaptive step-size mechanism based on the negative gradient vector.

Stability and steady-state performance have been investigated with the methods of mean square error using a few assumptions related to adaptive FIR coefficient and update control points LUT vectors. For the experimental simulation, the learning curves of mean square error of proposed AS-NLMS-HSAF algorithm is able to converge towards their steady-state and performance can reach the noise power at SNR = 40dB, despite varying 100-fold initial step-size parameters over 100 Monte Carlo trials. Moreover, the experiment trajectories of adaptive step-size parameters of the update coefficient FIR filter and control points LUT vectors are also converged to their equilibrium points for varying the initial step-size parameters in the steady-state.

In particular, Hammerstein structure can be interestingly modified in several applications of engineering fields as signal processing, data analysis in biomedical engineering and chemistry nonlinear processes.

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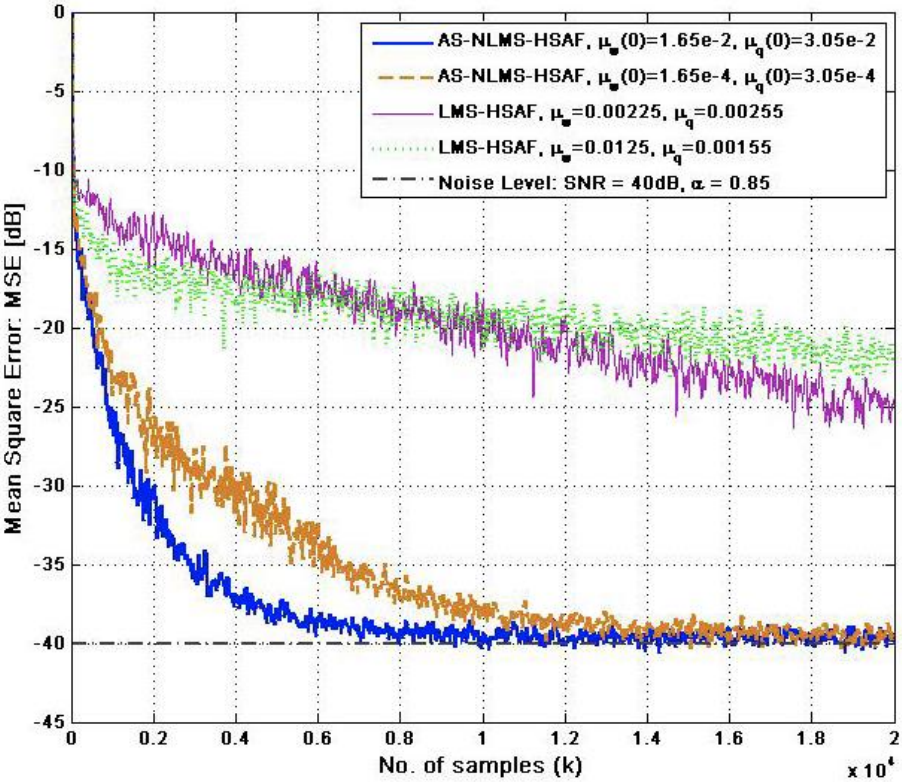

Figure.2 Comparison of experiment learning curves of adaptive HSAF-based algorithm based on mean square error for varying step-size parameters, when SNR = 40dB and α = 0.85.

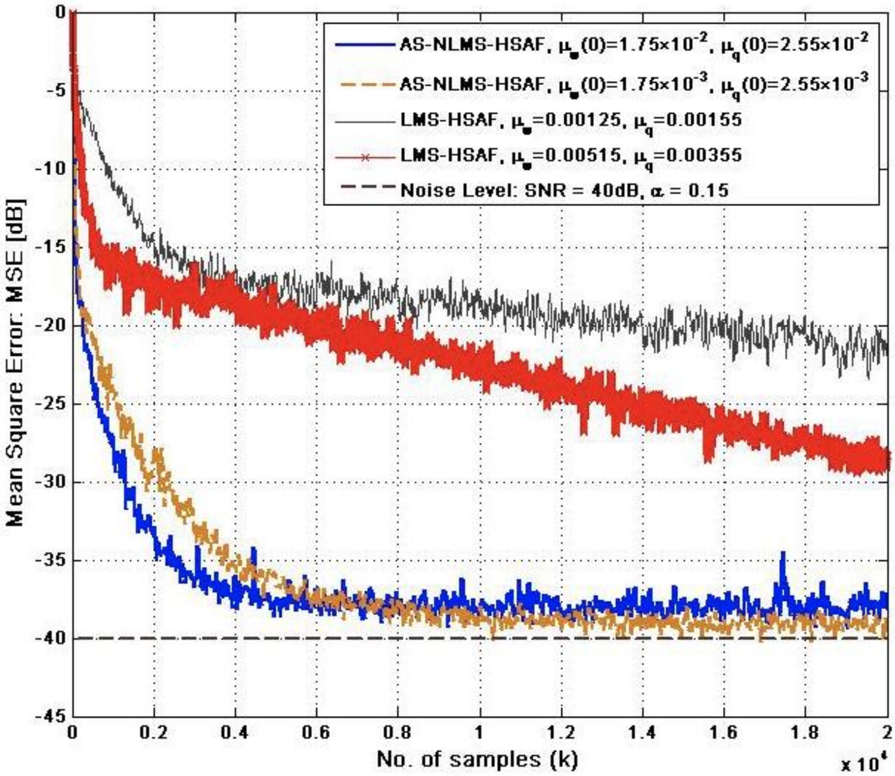

Figure.3 Comparison of experiment learning curves of adaptive HSAF-based algorithm based on mean square error for varying step-size parameters, when SNR = 40dB and α = 0.15.

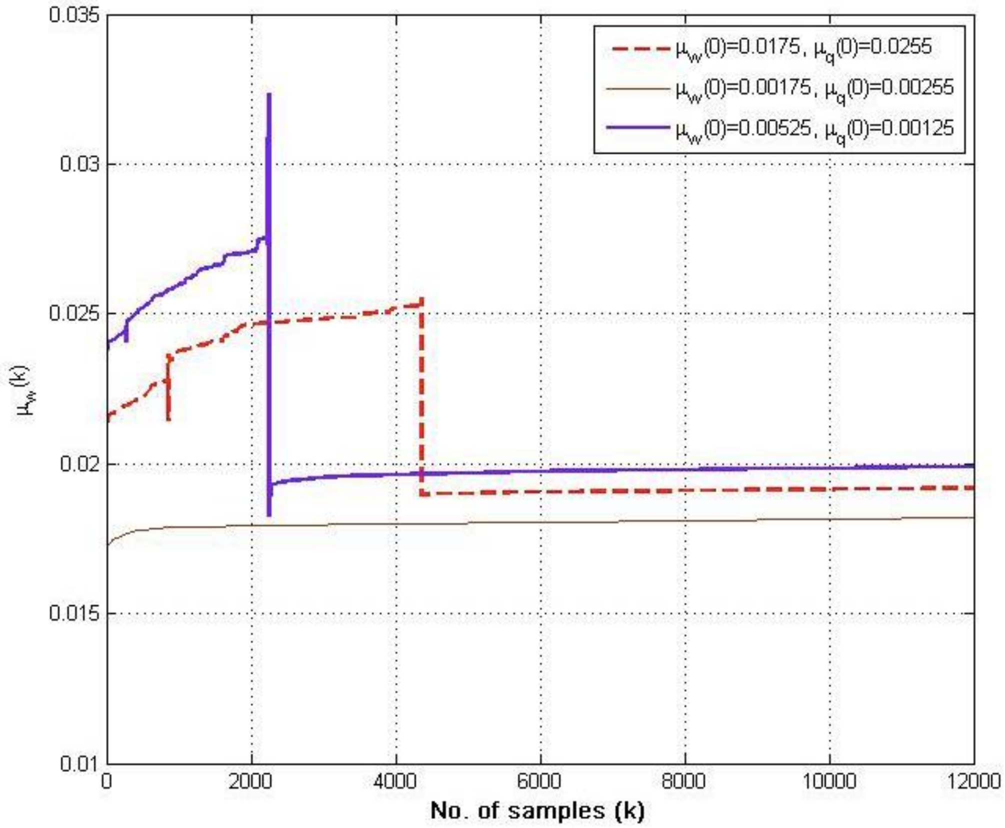

Figure.4 Approximations to the trajectories of update step-size $\mu_{\mathrm{w}}(k)$ of $\mathbf{w}(k)$ based on proposed AS-NLMS-HSAF algorithm with the varying initial condition.

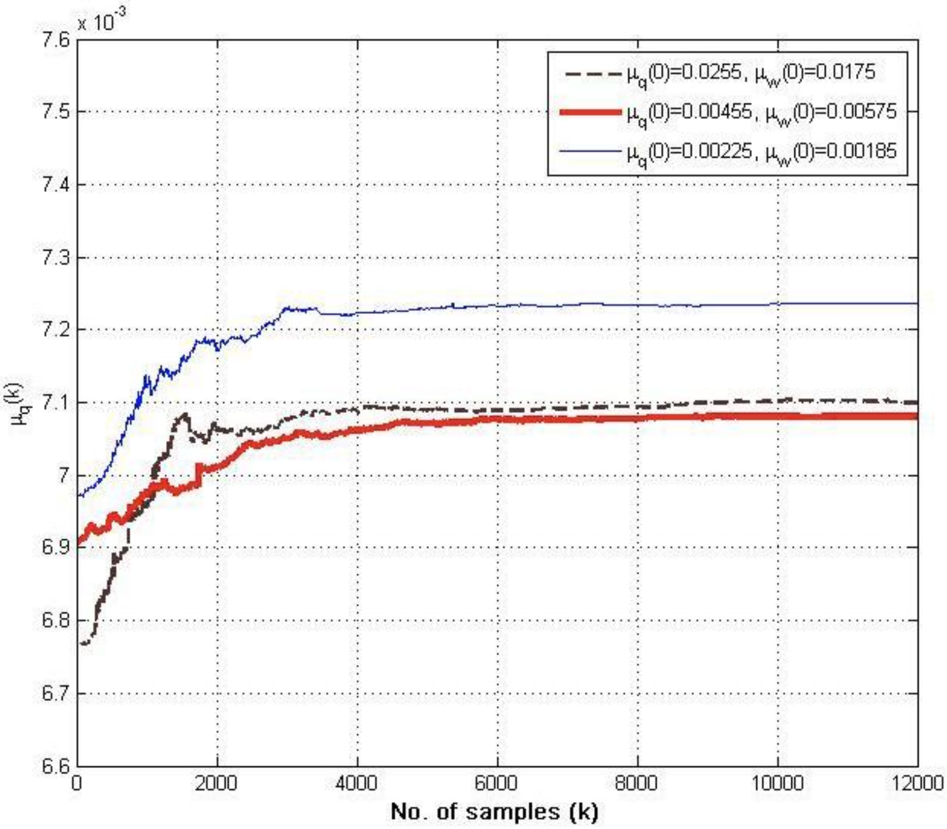

Figure.5 Approximations to the trajectories of update step-size $\mu_{\mathrm{q}}(k)$ of $\mathbf{q}_i(k)$ based on proposed AS-NLMS-HSAF algorithm with the varying initial condition.

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