Convergence and Stability Analysis of Spline Adaptive Filtering based on Adaptive Averaging Step-size Normalized Least Mean Square Algorithm

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Abstract: This paper presents a normalized version of least mean square algorithm with an adaptive averaging step-size mechanism to modify on autocorrelation between previous and present estimate error of system for updating step-size parameter. For achieving fast convergence, the proposed spline adaptive filter is combined with adaptive averaging step-size scheme and normalized version of least mean square approach. The convergence analysis and stability properties are accomplished. Simulation results of experiments depict that the trajectories of step-size parameters of the proposed algorithm converge to their own equilibria in spite of large variations in initial step-size settings. Proposed algorithm demonstrates more robust performance in mean square error and fast convergence compared with the conventional spline adaptive filter.

Keywords: Spline adaptive filtering, Nonlinear network, Normalized least mean square algorithm.

1. Introduction

Linear adaptive filtering is widely used for the solution to simply determine with the suitable constraint [1]. In opposition, many practical models necessitate the use of nonlinear adaptive filter in which nonlinear problem has more attention than linear operating system [2].

Spline adaptive filtering (SAF) based on least mean square (LMS) algorithm is a class of nonlinear adaptive filtering introduced in [2-4] with the low computation complexity and modelled in non-linear identification systems [5]. SAF is fabricated by adaptive linear finite impulse response (FIR) filtering followed by an adaptive lookup table (LUT).

Nonlinearity SAF structure using lookup table adjustment with the control points has been proposed in [3]. Sandwich SAF model in forms of cascade SAF architecture consists of a class of nonlinear models as linear-nonlinear-linear and nonlinear-linear-nonlinear models based on SAF structure, which can optimize using gradient-based condition in many conventional solutions of application [6, 7].

For nonlinear system identification, the authors in [8-10] conducted the normalized version of LMS (NLMS) scheme using the gradient-based criterion to improve performance of adaptive filtering, while the authors in [10] induced the potential performance in the case of infinite impulse response.

Against the impulsive noise, a set-membership scheme with the normalized version of least M-estimate algorithm has been developed in [11]. Simulation results depict that it can attain the achievable convergence rate. In [12], a sign normalized Wiener SAF is proposed in order to enhance the convergence by minimizing the absolute value of a posteriori error.
To establish the well tracking and fast convergence, the adaptive step-size mechanism based on LMS algorithm is a well-known approach with effective solution for achieving the convergence in linear adaptive filtering [13], [14]. In [15], an idea of time averaging applied on adaptive step-size algorithm for beam forming has been modified with the low computation. In [16], a low complexity step-size method by utilizing an approximate of autocorrelation error between present and previous estimate error is rearranged adaptively.

In this paper, we introduce a low complexity adaptive step-size approach based on normalized LMS algorithm in the SAF structure to achieve the fast convergence. In especial, we focus on the convergence analysis and mean square error performance of proposed algorithm.

This paper is arranged in this following. Section II describes briefly about SAF based on LMS. Section III proposes an adaptive averaging step-size algorithm for both the weight vectors of adaptive linear FIR filtering and interpolating control points of adaptive LUT by the minimizing cost function. Section IV shows the convergence and stability analysis of proposed algorithm. Experiment results and conclusion is in Section V and VI, respectively.

Notations are used through this paper. Operator \((\cdot)^T\) is the operation of transposition. Matrices and vectors are in bold uppercase and lowercase, respectively.

2. Spline adaptive filtering

The structure of spline adaptive filter (SAF), namely linear-nonlinear network, shows in Fig. 1. This network consists of linear and nonlinear part which is the linear part used the adaptive finite impulse response (FIR) filter and nonlinear part is an adaptive lookup table (LUT) with the spline interpolation network [2].

Consider a desired signal \(d_n\) as

\[
d_n = y_n + e_n
\]

where \(y_n\) is the spline adaptive filtering (SAF) output and \(e_n\) is the system error.

The output of adaptive FIR filter \(s_n\) can be defined as

\[
s_n = w_n^T x_n,
\]

where \(w_n\) is the adaptive tap-weight vector and \(x_n\) is the input vector as

\[
w_n = [ w_0 \ w_1 \ ... \ w_{N-1} ],
\]

\[
x_n = [ x_n \ x_{n-1} \ ... \ x_{n-N+1} ],
\]

Following [17], the output of SAF is defined as

\[
y_n = u_n^T q_{i,n} ,
\]

\[
u_n = [ u_{3,n}^1 \ u_{3,n}^2 \ u_{3,n} ]^T
\]

where \(q_{i,n}\) is the control points vector as
\[
q_{i,n} = [q_{i,n}, q_{i+1,n}, q_{i+2,n}, q_{i+3,n}]^T.
\]

Local parameter \(u_n\) and index \(i\) are defined as [18]

\[
u_n = \frac{s_n}{\Delta x}, \quad i = \left[\frac{s_n}{\Delta x} + \frac{q-1}{2}\right].
\]  

(5) 

where \(\Delta x\) is the uniform space between two-adjacent control points. \(Q\) is the number of control point, and \(\lfloor \cdot \rfloor\) is floor operator. The parameter \(s_n\) is concerned with a nonlinear activation function using the span index \(i\) and the local parameter \(u\), where \(u \in [0, 1]\). Spline basis matrix \(C\) is described in [2].

By minimizing cost function in the least mean square algorithm (LMS), we have [2]

\[
J(w_n, \text{q}_{\text{i},n}) = \frac{1}{2} \min_{w_n} \{ |e_n^2| \},
\]  

(6) 

where \(e_n\) is a priori estimation error \(e_n\) that arises from the model as

\[
e_n = d_n - y_n = d_n - u_n^T C q_{\text{i},n}.
\]  

(7) 

Hence, the adaptive tap-weight \(w_n\) and \(q_{\text{i},n}\) vectors take the specific form as

\[
w_{n+1} = w_n - \mu_w \frac{\partial J(w_n, q_{\text{i},n})}{\partial w_n},
\]  

(8) 

\[q_{\text{i},n+1} = q_{\text{i},n} - \mu_q \frac{\partial J(w_n, q_{\text{i},n})}{\partial q_{\text{i},n}},
\]  

(9) 

where \(\mu_w\) and \(\mu_q\) are the step-size parameters. Then, the gradient of the cost function in Eq. (6) is necessarily evaluated with respect to (w.r.t) the adaptive tap-weight \(w_n\) and \(q_{\text{i},n}\) vectors using the chain rule as

\[
\frac{\partial J(w_n, q_{\text{i},n})}{\partial w_n} = -e_n \frac{\partial y_n}{\partial u_n} \frac{\partial u_n}{\partial s_n} \frac{\partial s_n}{\partial w_n}
\]

\[\quad = -e_n \frac{\partial y_n}{\partial u_n} \frac{u_n^T C q_{\text{i},n} x_n}{\Delta x},
\]  

(10) 

where the derivative of \(u_n\) is given as

\[
u_n' = [3 u_n^2, 2 u_n, 1, 0],
\]  

(11) 

and

\[
\frac{\partial J(w_n, q_{\text{i},n})}{\partial q_{\text{i},n}} = -e_n \frac{\partial y_n}{\partial u_n} \frac{\partial u_n}{\partial s_n} \frac{\partial s_n}{\partial q_{\text{i},n}}
\]

\[\quad = -e_n \frac{\partial y_n}{\partial u_n} C^T u_n.
\]  

(12) 

According to Eqs. (10) and (12), the tap-weight LMS \(w_n\) and \(q_{\text{i},n}\) vectors in the recursion form can be represented as [2]

\[
w_{n+1} = w_n + \mu_w u_n^T C q_{\text{i},n} x_n e_n,
\]  

(13) 

\[
q_{\text{i},n+1} = q_{\text{i},n} + \mu_q C^T u_n e_n,
\]  

(14) 

where \(\mu_w\) and \(\mu_q\) are the fixed step-size parameters for tap-weight \(w_n\) and for the control points \(q_{\text{i},n}\), which incorporate with the other constant.

3. Proposed adaptive averaging step-size normalized least mean square algorithm for spline adaptive filtering

Following [11], the minimized cost function of normalized least mean square algorithm for SAF is expressed as

\[
J(w_n, q_{\text{i},n}) = \frac{1}{2} \min_{w_n} \{ (u_n^T u_n)^{-1} |e_n^2| \}
\]  

(15) 

where \(e_n\) is defined in (7).

And the update tap-weight estimated vector \(w_n\) at symbol \(n\) can be expressed by

\[
w_{n+1} = w_n - \mu_w \frac{\partial J(w_n, q_{\text{i},n})}{\partial w_n}.
\]  

(16) 

By differentiating the cost function in Eq. (15) w.r.t \(w_n\) with the chain rule, that is

\[
\frac{\partial J(w_n, q_{\text{i},n})}{\partial w_n} = (u_n^T u_n)^{-1} \left( -e_n \frac{\partial y_n}{\partial u_n} \frac{\partial u_n}{\partial s_n} \frac{\partial s_n}{\partial w_n} \right)
\]

\[\quad = (u_n^T u_n)^{-1} \left[ -e_n \frac{\partial y_n}{\partial u_n} C q_{\text{i},n} x_n \right].
\]  

(17) 

Finally, we introduce the proposed tap-weight estimated vector \(w_n\) of adaptive FIR filter based on normalized least mean square algorithm obtained by

\[
w_{n+1} = w_n + \mu_{w_n} u_n' C q_{\text{i},n} x_n e_n \frac{\partial y_n}{\partial u_n} \frac{\partial u_n}{\partial s_n} \frac{\partial s_n}{\partial w_n}.
\]  

(18) 

where \(\mu_{w_n}\) is the adaptive step-size parameter for learning rate of linear part of SAF structure.

Similarly, the update estimated control points vector \(q_{\text{i},n}\) at symbol \(n\) can be obtained by

\[
q_{\text{i},n+1} = q_{\text{i},n} - \mu_{q_{\text{i},n}} \frac{\partial J(w_n, q_{\text{i},n})}{\partial q_{\text{i},n}}.
\]  

(19)
Hence, the gradient of the cost function in Eq. (15) w.r.t \( q_{in} \) using the chain rule is defined by

\[
\frac{\partial J(w_n, q_{in})}{\partial q_{in}} = (u_n^T u_n)^{-1}\left\{-\frac{\partial y_n}{\partial u_n} \frac{\partial u_n}{\partial s_n} \frac{\partial s_n}{\partial q_{in}}\right\}
\]

\[
= (u_n^T u_n)^{-1}\{-e_n C^T u_n\}. \tag{20}
\]

Therefore, we present the control points vector \( q_{in} \) based on normalized least mean square algorithm of nonlinear network in the adaptive lookup table as

\[
\triangle q_{i,n+1} = q_{i,n} + \mu_{q_n} \frac{C r u_n e_n}{u_n^T u_n}. \tag{21}
\]

where \( \mu_{q_n} \) is the adaptive step-size parameter for nonlinear part of SAF structure.

3.1 Adaptive averaging step-size algorithm for spline adaptive filtering

The main objective of adaptive averaging step-size mechanism is to improve as follows. Following [17], if the estimate error is far off the optimal value, the step-size parameter will be increased. Meanwhile, the estimate error is near the optimum, the step-size parameter will be decreased automatically.

The proposed idea is to average step-size parameter with autocorrelation of previous and present estimate error of network system for update \( \mu_w \) and \( \mu_{q_n} \) adaptively.

Therefore, we modify the adaptive averaging step-size \( \mu_w \) of tap-weight \( w_n \) vector concerning with the estimation of an averaging of autocorrelation \( \{e_{n-1}^2\} \) as

\[
\mu_w = \alpha_w \cdot \mu_{w,n-1} + \beta_w \cdot |e_n|^2, \tag{22}
\]

\[
\xi_n = \gamma : \xi_{n-1} + (1-\gamma)\{e_{n-1}^2\}, \tag{23}
\]

where \( \beta_w \) is a scaled variable for prediction error, \( \gamma \) is close to 1 and \( 0 < \alpha_w < 1 \).

We note that there are two reasons related with \( \xi_n \) are as follows. First, the autocorrelation of error is generally measured for optimal performance. Second, the uncorrelated noise sequence is rejected on the update step-size mechanism.

3.2 Modified adaptive step-size algorithm

Following [19], the learning rate of step-size is controlled by squared estimate error. If an error is large, the step-size parameter will increase. While a small error will yield misadjustment with the decreased step-size value. Therefore, the step-size parameter \( \mu_{q_n} \) of control points vector \( q_{in} \) is

\[
\mu_{q_n} = \alpha_q \cdot \mu_{q_{n-1}} + \beta_q \cdot |e_n|^2, \tag{24}
\]

where \( 0 < \alpha_q < 1, \beta_q > 0 \) and a priori estimate error \( e_n \) is given in Eq. (7).

Summary of proposed adaptive averaging step-size mechanism based on the normalized version of least mean square algorithm for spline adaptive filter (AAS-NLMS-SAF) is shown in Table 1.

4. Convergence and stability analysis

In order to achieve optimal performance, we determine an adaptive learning rate that minimizes the instantaneous output error of filter by performing Taylor series expansion of error \( e_n \). The approach intends to the optimal learning rate to ensure the convergence at the steady-state.

4.1 Convergence analysis of proposed algorithm

Convergence properties of adaptive tap-weight \( w_n \) vector can be determined by using Taylor series expansion of estimate error \( e_n \) as [2]

\[
e_{n+1} = e_n + \frac{\partial e_n}{\partial w_n} \Delta w_n, \tag{25}
\]

where an estimate error \( e_n \) is given as

\[
e_n = d_n - u_n^T C q_{in}. \tag{26}
\]

Differentiating \( e_n \) in Eq. (26) w.r.t \( w_n \) with the chain rule, we get

\[
\frac{\partial e_n}{\partial w_n} = -u_n^T C q_{in} x_n. \tag{27}
\]

where \( u'_n \) is given in Eq. (11).

From Eq. (18), we have the change of \( w_n \) as

\[
\Delta w_n = w_{n+1} - w_n = \frac{\mu_w u_n^T C q_{in} x_n e_n}{\Delta x(u_n^T u_n)}. \tag{28}
\]

By substituting (27) and (28) into (25), we arrive at

\[
e_{n+1} = e_n - \mu_w \frac{u_n^T C q_{in} x_n e_n}{\Delta x(u_n^T u_n)}. \tag{29}
\]

where \( \varnothing_n \) is given by

\[
\varnothing_n = u'_n C q_{in}. \tag{30}
\]
Table 1. Proposed spline adaptive filtering based on the adaptive averaging step-size normalized least mean square algorithm (AAS-NLMS-SAF)

<table>
<thead>
<tr>
<th>Initialize: ( \mathbf{w}(0) = \varphi_w \cdot [1 \ 0 \ldots \ 0]^T ), ( \mathbf{q}(0) = [1 \ 0 \ldots \ 0]^T ),</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{C} = \frac{1}{2} \begin{bmatrix} -1 &amp; 3 &amp; -3 &amp; 1 \ 2 &amp; -5 &amp; 4 &amp; -1 \ -1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 2 &amp; 0 &amp; 0 \end{bmatrix} )</td>
</tr>
<tr>
<td>for ( n = 0, 1, 2, \ldots, N - 1 ).</td>
</tr>
<tr>
<td>1) Calculate the output of adaptive FIR filter ( s_n ),</td>
</tr>
<tr>
<td>( s_n = \mathbf{w}_n^T \mathbf{x}_n ),</td>
</tr>
<tr>
<td>2) Compute the local parameter ( u_n ) and index ( i ) as</td>
</tr>
</tbody>
</table>
| \[
| u_n = \frac{s_n}{\Delta x} - \frac{s_n}{\Delta x} \|
| i = [\frac{s_n}{\Delta x} + \frac{q-1}{2}]. |
| 3) Calculate the error \( e_n \) as |
| \[
| e_n = d_n - u_n^T \mathbf{C} \mathbf{q}_{n,n} |
| 4) Compute the adaptive averaging step-size \( \mu_{w_n} \) of \( \mathbf{w}_n \), |
| \[
| \mu_{w_n} = \alpha_w \cdot \mu_{w_{n-1}} + \beta_w \cdot |\xi_n|^2 |
| \[
| \xi_n = \gamma \cdot \xi_{n-1} + (1 - \gamma) \{e_{n-1} \cdot e_n\}, |
| 5) Calculate the modified step-size \( \mu_{q_{n,n}} \) of \( \mathbf{q}_{n,n} \), |
| \[
| \mu_{q_n} = \alpha_q \cdot \mu_{q_{n-1}} + \beta_q \cdot |e_n|^2 |
| 6) Determine the tap-weight vector \( \mathbf{w}_n \) and the control points vector \( \mathbf{q}_{n,n} \) as |
| \[
| \mathbf{w}_{n+1} = \mathbf{w}_n + \mu_{w_n} \frac{\mathbf{u}_n^T \mathbf{C} \mathbf{q}_{n,n} \mathbf{x}_n e_n}{\Delta x \ |\mathbf{u}_n|^2 |\mathbf{u}_n|}, |
| \[
| \mathbf{q}_{n+1} = \mathbf{q}_{n,n} + \mu_{q_n} \frac{\mathbf{c}_n^T \mathbf{u}_n \mathbf{e}_n}{\mathbf{u}_n^T \mathbf{u}_n}, |
| end |

Therefore, the estimate error can be rewritten as |
| \[
| e_{n+1} = \left[1 - \frac{\mu_{w_n}}{(\Delta x)^2} \left(\frac{\xi_n^2 |\mathbf{x}_n|^2}{|\mathbf{u}_n^T \mathbf{u}_n|}\right)\right] \cdot |e_n| \quad (31) |

Taking the norm of both sides in (31), we have |
| \[
| |e_{n+1}| = \left|1 - \frac{\mu_{w_n}}{(\Delta x)^2} \left(\frac{\xi_n^2 |\mathbf{x}_n|^2}{|\mathbf{u}_n^T \mathbf{u}_n|}\right)\right| \cdot |e_n|. \quad (32) |

Therefore, the proposed step-size \( \mu_{w_n} \) of tap-weight vector \( \mathbf{w}_n \) in the adaptive FIR filter reaches |
| \[
| \mu_{w_n} \approx \frac{2(\mathbf{u}_n^T \mathbf{u}_n)^2}{\xi_n^2 |\mathbf{x}_n|^2}, \quad (33) |
| where we assume that \( |e_{n+1}| < |e_n| \). |

Similarly, we determine a bound on \( \mu_{q_n} \) with the Taylor series expansion of estimate error \( e_n \) as |
| \[
| e_{n+1} = e_n + \frac{\partial e_n}{\partial \mathbf{q}_{n,n}} \cdot \Delta \mathbf{q}_{n,n}, \quad (34) |
| where the derivative of \( e_n \) w.r.t \( \mathbf{q}_{n,n} \) is given by |
| \[
| \frac{\partial e_n}{\partial \mathbf{q}_{n,n}} = -\frac{e_n^T \mathbf{u}_n}{\mathbf{u}_n^T \mathbf{u}_n}, \quad (35) |
| And From (21), we have the change of \( \mathbf{q}_{n,n} \) as |
| \[
| \Delta \mathbf{q}_{n,n} = \mu_{q_n} \frac{\mathbf{c}_n^T \mathbf{u}_n \mathbf{e}_n}{\mathbf{u}_n^T \mathbf{u}_n}, \quad (36) |
| Hence, we substitute Eqs. (35) and (36) into Eq. (34), we have |
| \[
| e_{n+1} \approx \left[1 - \mu_{q_n} \frac{\mathbf{c}_n^T \mathbf{u}_n \mathbf{e}_n}{\mathbf{u}_n^T \mathbf{u}_n}\right] \cdot |e_n|. \quad (37) |
| Taking the norm of both sides in Eq. (37), we get |
| \[
| |e_{n+1}| \approx \left|1 - \mu_{q_n} \frac{\mathbf{c}_n^T \mathbf{u}_n \mathbf{e}_n}{\mathbf{u}_n^T \mathbf{u}_n}\right| \cdot |e_n|. \quad (38) |
| Therefore, the adaptive learning rate \( \mu_{q_n} \) becomes |
| \[
| \mu_{q_n} \approx \frac{2 \mathbf{u}_n^T \mathbf{u}_n}{\mathbf{c}_n^T \mathbf{c}_n}. \quad (39) |

4.2 Mean square error performance of proposed algorithm

In this section, we consider the mean square error performance at steady-state in the derivation of excess mean square error (EMSE) of nonlinear adaptive FIR filter and the control points vector in the adaptive LUT.

Following [6], we determine the \( \varepsilon_n \) is a priori error of system, \( \varepsilon_{w_n} \) is a priori error concerned the tap-weight vector \( \mathbf{w}_n \) and \( \varepsilon_{q_n} \) is a priori error involved the control points vector \( \mathbf{q}_{n,n} \).

To encourage the analysis, the proposed adaptive averaging step – size normalized least
mean square (AAS - NLMS) algorithm is under a few assumptions.

**Assumption 1:** We consider that the noise sequence of system \( \eta_n \) is independent and identically distributed with variance of noise \( \sigma^2 \) and zero mean.

**Assumption 2:** We consider that the noise sequence of system \( \eta_n \) is independent of \( x_n, s_n, \varepsilon_n, \varepsilon_{wn}, \) and \( \varepsilon_qn \).

Let us assume the estimate weight noise vector \( \eta_{wn} \) concerned with the tap – weight vector \( w_n \) as

\[
\eta_{wn} = w_0 - w_n.
\]  

where \( \eta_{wn} = [\eta_{wn1} \ldots \eta_{wnN-1}] \).

From Eq. (18), we can write the update weight noise vector \( \eta_{wn+1} \) as

\[
\eta_{wn+1} = \eta_{wn} - (w_{n+1} - w_n) - \mu_n \frac{\eta_n x_n \varepsilon_n}{\Delta x (u_n^T u_n)}.
\]  

where \( \Phi_n \) is given in Eq. (30).

To evaluate the square of update weight noise vector \( \| \eta_{wn} \|^2 \) of Eq. (41), we obtain

\[
\| \eta_{wn+1} \|^2 = \| \eta_{wn} \|^2 - \mu_n \frac{\eta_n x_n \varepsilon_n}{\Delta x (u_n^T u_n)} + \mu_n \frac{\eta_n x_n \varepsilon_n}{\Delta x (u_n^T u_n)}.
\]  

**Assumption 3:** We consider the condition necessary for the convergence of mean, that is

\[
E\{\| \eta_{wn} \|^2 \} = E\{\| \eta_{wn} \|^2 \} \text{ as } n \to \infty.
\]

From Assumption (3), the update \( \eta_{wn} \) in (42) can be rewritten as

\[
2\eta_{wn} \cdot w_{n+1} - \mu_n \frac{\eta_n x_n \varepsilon_n}{\Delta x (u_n^T u_n)} = \mu_n \frac{\eta_n x_n \varepsilon_n}{\Delta x (u_n^T u_n)} + \mu_n \frac{\eta_n x_n \varepsilon_n}{\Delta x (u_n^T u_n)}
\]

where \( E_{wn} \) is given by

\[
E_{wn} = \eta_{wn} x_n.
\]

To redefine the \textit{a priori} error of system \( e_n \) as

\[
e_n = E_{wn} + \eta_{wn}.
\]

Taking the expectation onto the noise in Eqs. (44) and (45) with the condition at steady-state for \( n \) closes to infinity, we get

\[
E\{E_{wn} \cdot e_n \} = E\{E_{wn}(E_{wn} + \eta_{wn})\} = E\{E_{wn}^2\} (46)
\]

and

\[
E\{e_n^2\} = E\{(E_{wn} + \eta_{wn})^2\}
\]

\[
= E\{E_{wn}^2 + 2E_{wn}\eta_{wn} + \eta_{wn}^2\}
\]

\[
\approx E\{E_{wn}^2 + \xi_{wn}^2\}. (47)
\]

where \( \xi_{wn}^2 \) is the minimum MSE involved with \( w_n \).

Substituting Eqs. (46) and (47) into Eq. (43), we have

\[
2E\{E_{wn}^2\} = \frac{\mu_n}{\Delta x} \left[ \frac{\eta_n x_n \varepsilon_n}{(u_n^T u_n)} \cdot E\{E_{wn}^2\} + \xi_{wn}^2 \right]
\]

\[
\left[ 2 - \frac{\mu_n}{\Delta x} \frac{\eta_n x_n \varepsilon_n}{(u_n^T u_n)} \right] \left[ E_{wn}^2 \right] = \frac{\mu_n}{\Delta x} \frac{\eta_n x_n \varepsilon_n}{(u_n^T u_n)} \cdot E\{E_{wn}^2\}
\]

\[
E\{E_{wn}^2\} = \frac{\mu_n}{\Delta x} \frac{\eta_n x_n \varepsilon_n}{(u_n^T u_n)} \cdot E\{E_{wn}^2\}
\]

If \( \mu_n \) is very small, we have

\[
\xi_w = E\{E_{wn}^2\} = \frac{\mu_n}{\Delta x} \frac{\eta_n x_n \varepsilon_n}{(u_n^T u_n)} \cdot E\{E_{wn}^2\}
\]

where \( \xi_w \) is the excess MSE concerned with \( w_n \).

In a similar manner, we assume that the noise sequence of estimated weight noise vector \( \eta_{qn} \) involved with the control points vector \( q_{in} \) as

\[
\eta_{qn} = q_0 - q_{in}.
\]

where \( \eta_{qn} = [\eta_{qn0} \eta_{qn1} \ldots \eta_{qnn-1}] \).

From Eq. (21), the update weight noise vector \( \eta_{qn} \) can be expressed as

\[
\eta_{qn+1} = \eta_{qn} - \mu_{qn} \frac{w_n^T e_n}{(u_n^T u_n)}.
\]

Then, we evaluate the square of noise vector \( \| \eta_{qn} \|^2 \) using Eq. (51), that is

\[
\| \eta_{qn+1} \|^2 = \| \eta_{qn} \|^2 - 2\eta_{qn} \cdot \mu_{qn} \frac{w_n^T e_n}{(u_n^T u_n)} + \mu_{qn} \frac{w_n^T e_n}{(u_n^T u_n)} (52)
\]

**Assumption 4:** We regard that

\[
E\{\| \eta_{qn+1} \|^2 \} = E\{\| \eta_{qn} \|^2 \} \text{ as } n \to \infty
\]
From Assumption (4), the update $\eta_{q_n}$ in Eq. (52) can be calculated as

$$
2\eta_{q_n} \cdot \mu_{q_n} \cdot \mu_{q_n} \cdot \eta_{q_n} \cdot (u_n^T u_n) = \frac{\mu_{q_n} \| u^T e \|^2 e_{x_n}^2}{(u_n^T u_n)^2}.
$$

(53)

where $e_{x_n}$ is given as

$$
e_{x_n} = E_{x_n} + \eta_{x_n}.
$$

(54)

So, we determine that the a priori error $e_n$ is involved with $q_{l,n}$ as

$$
e_n = E_{q_n} + \eta_{q_n}.
$$

(55)

Taking the expectation into the noise in Eq. (53) and (55) at steady-state for $n \to \infty$, we have

$$
E\{e_{x_n} \cdot e_n\} = E\{E_{x_n} \cdot (E_{q_n} + \eta_{q_n})\} = E\{E_{q_n}^2\},
$$

(56)

$$
E\{E_{q_n}^2\} = \{E_{q_n}^2 + \eta_{q_n}^2\} \approx E\{E_{q_n}^2 + \xi_{q_n}^2\},
$$

(57)

where $\xi_{q_n}^2$ is the minimum MSE involved with $q_{l,n}$. Replacing Eqs. (56) and (57) into Eq. (53), we get

$$
2E\{E_{q_n}^2\} = \mu_{q_n} \| u^T C \|^2 E\{E_{q_n}^2 + \xi_{q_n}^2\}.
$$

(58)

If $\mu_{q_n}$ is very small, we get

$$
\zeta_q \approx E\{E_{q_n}^2\} = \frac{\mu_{q_n} \| u^T e \|^2 E\{E_{q_n}^2\}}{2(u_n^T u_n)}.
$$

(59)

where $\zeta_q$ is the excess MSE concerned with $q_{l,n}$.

5. Experimental results

In this section, we provide the experimental tests in system identification by simulating the random process. The input coloured signal for all experiments comprises 5,000 samples of the signal generated in the system identification over 100 Monte Carlo trials by following [20].

$$
x_n = \alpha \cdot x_{n-1} + \sqrt{1 - \alpha^2} \cdot \psi_n,
$$

(60)

where $\psi_n$ denotes as a zero mean white Gaussian noise with unitary variance and $0.1 \leq \alpha \leq 0.99$.

![Learning curves of $\mu_w(n)$ of tap-weight $w_n$ vector of proposed AAS-NLMS-SAF algorithm with the different $\alpha = 0.1, 0.25, 0.75$ and SNR = 40dB](image-url)

Figure 3 Learning curves of $\mu_q(n)$ of control points $q_{i,n}$ vector of proposed AAS-NLMS-SAF algorithm the different $\alpha = 0.1, 0.25, 0.75$ and SNR = 40dB

Figure 4 Mean square error (MSE) of proposed ASS-NLMS-SAF algorithm compare with LMS-SAF [20] with the different of initial step-size parameter using SNR = 40dB and $\alpha = 0.10$
We consider the mean square error (MSE) computed in dB as

$$\text{MSE}_n = 10 \log \left( E \left\{ (d_n - u_n^T C q_n)^2 \right\} \right). \quad (61)$$

A 23-point LUT $q_0$ is implemented for a nonlinear memoryless target function that is interpolated with a uniform third degree spline and SAF model is used as $\Delta x = 0.2$ [4] and $C$ is a Catmul-Rom spline as described in [2].

Initial parameters of all SAF model are as follows: $w(0) = q_0 \cdot [1, 0, ..., 0]^T$, where $\varphi_w = 1 \times 10^{-3}$, $q_0(0) = [1, 0, ..., 0]^T$, SNR = 40dB, length of filter is 5. For initial parameters for spline adaptive filtering based on least mean square (LMS-SAF) algorithm [20] are as: $\mu_w = \mu_q = 0.025, 0.035, 0.050$. Summary of LMS-SAF is shown in Table 2.

Other parameters for proposed AAS-NLMS-SAF algorithm are as: $\mu_w(0) = \mu_q(0) = 1.5 \times 10^{-4}, 1.5 \times 10^{-2}, 2.5 \times 10^{-2}, 3.5 \times 10^{-2}, 5.5 \times 10^{-2}$. The fixed parameters are as follows: $\alpha_w = \alpha_q = 0.975$, $\beta_w = 2.95 \times 10^{-3}$, $\beta_q = 1.95 \times 10^{-3}$, $\gamma = 0.97$.

Learning rates of step-size parameters $\mu_w$ of tap-weight vector and $\mu_q$ of control point vector of proposed AAS-NLMS-SAF algorithm are shown in Figs. 2 and 3 with the different initial parameter of $\mu_w(0), \mu_q(0)$ at SNR = 40dB with the different $\alpha$ in (60) generated the input coloured signal. It is seen that both learning curves of $\mu_w$ and $\mu_q$ converge to their equilibria despite 100-fold of initial step-size situations at steady-state.

In terms of MSE performance, simulation results shown for the proposed experiments with the two choices of parameter $\alpha = 0.10, 0.95$ which are presented in Fig. 4 and Fig. 5, respectively. At steady-state, the performance of proposed AAS-NLMS-SAF algorithm closes to the noise power. In
addition, we notice that the performance of proposed AAS-NLMS-SAF algorithm outperforms to converge faster and robust mechanism when compared with the LMS-SAF algorithm using the variants of fixed step-size parameter.

6. Conclusion

In this paper, we propose a step-size approach in term of averaging of square error for spline adaptive filtering (AAS-NLMS-SAF). We describe how to derive the proposed adaptive averaging step-size algorithm with the method of normalised version of LMS algorithm on spline adaptive filtering. By using an estimation of autocorrelation between present estimated error and a priori estimated error, the adaptive averaging step-size scheme is proposed on SAF. The convergence and stability analysis of proposed AAS-NLMS-SAF algorithm examine in terms of mean square error and excess mean square error concerned with adaptive tap-weight FIR vector and control points vector in the adaptive LUT.

Both the trajectories of adaptive step-size parameters can converge into each equilibrium in spite of 100-fold initial variations. Learning curves of MSE performance are illustrated to converge dramatically to steady-state in comparison with the existing LMS-SAF algorithm using the fixed step-size parameters.

Especially, SAF can perform well with low-cost complexity beside the existing FIR structures. Because of the recursion form, SAF can be modified in many practical cases such as nonlinear channel equalization, biomedical data analysis and control applications.

References


